EXAMPLE 6 Evaluate $\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$.

Solution

$$\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = [\sin^{-1} x]_0^{1/2} = \sin^{-1}\frac{1}{2} - \sin^{-1} 0$$

$$= \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

EXAMPLE 7 A man standing on top of a vertical cliff is 200 feet above a lake. As he watches, a motorboat moves directly away from the foot of the cliff at a rate of 25 feet per second. How fast is the angle of depression of his line of sight changing when the boat is 150 feet from the foot of the cliff?

Solution The essential details are shown in Figure 8. Note that θ, the angle of depression, is

$$\theta = \tan^{-1}\left(\frac{200}{x}\right)$$

Thus,

$$\frac{d\theta}{dt} = \frac{1}{1 + (200/x)^2} \cdot \frac{-200}{x^2} \cdot \frac{dx}{dt} = \frac{-200}{x^2 + 40{,}000} \cdot \frac{dx}{dt}$$

When we substitute $x = 150$ and $dx/dt = 25$, we obtain $d\theta/dt = -0.08$ radians per second.

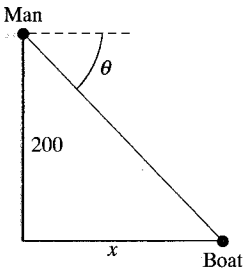


Figure 8

Concepts Review

1. To obtain an inverse for the sine function, we restrict its domain to _____. The resulting inverse function is denoted by $\sin^{-1}$ or by _____.

2. To obtain an inverse for the tangent function, we restrict the domain to _____. The resulting inverse function is denoted by $\tan^{-1}$ or by _____.

3. $D_x \sin(\arcsin x) =$ _____.

4. Since $D_x \arctan x = 1/(1 + x^2)$, it follows that $4\int_0^1 1/(1 + x^2)\,dx =$ _____.

Problem Set 7.7

In Problems 1–10, find the exact value without using a calculator.

1. $\arccos\left(\frac{\sqrt{2}}{2}\right)$

2. $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$

3. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

4. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

5. $\arctan(\sqrt{3})$

6. $\operatorname{arcsec}(2)$

7. $\arcsin(-\frac{1}{2})$

8. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

9. $\sin(\sin^{-1} 0.4567)$

10. $\cos(\sin^{-1} 0.56)$

C *In Problems 11–18, approximate each value.*

11. $\sin^{-1}(0.1113)$

12. $\arccos(0.6341)$

13. $\cos(\operatorname{arccot} 3.212)$

14. $\sec(\arccos 0.5111)$

15. $\sec^{-1}(-2.222)$

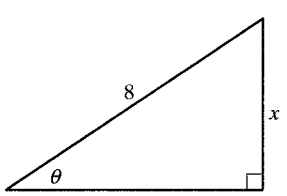
16. $\tan^{-1}(-60.11)$

17. $\cos(\sin(\tan^{-1} 2.001))$

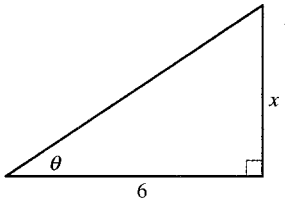
18. $\sin^2(\ln(\cos 0.5555))$

In Problems 19–24, express θ in terms of x using the inverse trigonometric functions $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, and $\sec^{-1}$.

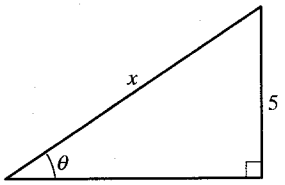
19.



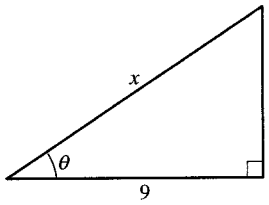
20.



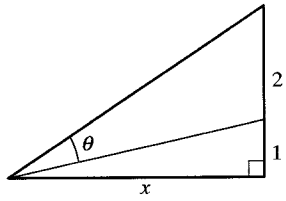
21.



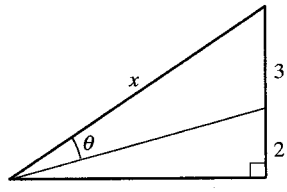
22.



23.



24.



In Problems 25–28, find each value without using a calculator (see Example 4).

25. $\cos\left[2\sin^{-1}\left(-\frac{2}{5}\right)\right]$ **26.** $\tan\left[2\tan^{-1}\left(\frac{1}{3}\right)\right]$

27. $\sin\left[\cos^{-1}\left(\frac{3}{5}\right)+\cos^{-1}\left(\frac{5}{13}\right)\right]$

28. $\cos\left[\cos^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{12}{13}\right)\right]$

In Problems 29–32, show that each equation is an identity.

29. $\tan(\sin^{-1}x)=\dfrac{x}{\sqrt{1-x^2}}$

30. $\sin(\tan^{-1}x)=\dfrac{x}{\sqrt{1+x^2}}$

31. $\cos(2\sin^{-1}x)=1-2x^2$

32. $\tan(2\tan^{-1}x)=\dfrac{2x}{1-x^2}$

33. Find each limit.

(a) $\lim_{x\to\infty}\tan^{-1}x$ (b) $\lim_{x\to-\infty}\tan^{-1}x$

34. Find each limit.

(a) $\lim_{x\to\infty}\sec^{-1}x$ (b) $\lim_{x\to-\infty}\sec^{-1}x$

35. Sketch the graph of $y=\cot^{-1}x$, assuming that it has been obtained by restricting the domain of the cotangent to $(0,\pi)$.

In Problems 36–47, find dy/dx.

36. $y=e^{\tan x}$ **37.** $y=\ln(\sec x+\tan x)$

38. $y=-\ln(\csc x+\cot x)$ **39.** $y=\sin^{-1}(2x^2)$

40. $y=\arccos(e^x)$ **41.** $y=x^3\tan^{-1}(e^x)$

42. $y=e^x\arcsin x^2$ **43.** $y=(\tan^{-1}x)^3$

44. $y=\tan(\cos^{-1}x)$ **45.** $y=\sec^{-1}(x^3)$

46. $y=(\sec^{-1}x)^3$ **47.** $y=(1+\sin^{-1}x)^3$

In Problems 48–58, evaluate each integral.

48. $\displaystyle\int x\sin(x^2)\,dx$ **49.** $\displaystyle\int\sin 2x\cos 2x\,dx$

50. $\displaystyle\int\tan x\,dx=\int\frac{\sin x}{\cos x}\,dx$ **51.** $\displaystyle\int_0^1 e^{2x}\cos(e^{2x})\,dx$

52. $\displaystyle\int_0^{\pi/2}\sin^2 x\cos x\,dx$ **53.** $\displaystyle\int_0^{\sqrt{2}/2}\frac{1}{\sqrt{1-x^2}}\,dx$

54. $\displaystyle\int_{\sqrt{2}}^2\frac{dx}{x\sqrt{x^2-1}}$ **55.** $\displaystyle\int_{-1}^1\frac{1}{1+x^2}\,dx$

56. $\displaystyle\int_0^{\pi/2}\frac{\sin\theta}{1+\cos^2\theta}\,d\theta$ **57.** $\displaystyle\int\frac{1}{1+4x^2}\,dx$

58. $\displaystyle\int\frac{e^x}{1+e^{2x}}\,dx$

C **59.** A picture 5 feet in height is hung on a wall so that its bottom is 8 feet from the floor, as shown in Figure 9. A viewer with eye level at 5.4 feet stands b feet from the wall. Express θ, the vertical angle subtended by the picture at her eye, in terms of b, and then find θ if $b=12.9$ feet.

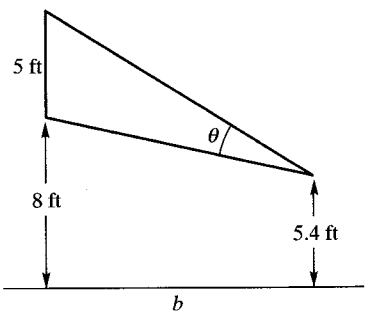


Figure 9

60. Find formulas for $f^{-1}(x)$ for each of the following functions f, first indicating how you would restrict the domain so that f has an inverse. For example, if $f(x)=3\sin 2x$ and we restrict the domain to $-\pi/4\le x\le\pi/4$, then $f^{-1}(x)=\frac{1}{2}\sin^{-1}(x/3)$.

(a) $f(x)=3\cos 2x$ (b) $f(x)=2\sin 3x$

(c) $f(x)=\frac{1}{2}\tan x$ (d) $f(x)=\sin\dfrac{1}{x}$

61. By repeated use of the addition formula

$$\tan(x+y)=(\tan x+\tan y)/(1-\tan x\tan y)$$

show that

$$\frac{\pi}{4}=3\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{5}{99}\right)$$

62. Verify that

$$\frac{\pi}{4}=4\tan^{-1}\left(\frac{1}{5}\right)-\tan^{-1}\left(\frac{1}{239}\right)$$

a result discovered by John Machin in 1706 and used by him to calculate the first 100 decimal places of π.

Problem Set 7.7

1. $\frac{\pi}{4}$ **3.** $-\frac{\pi}{3}$ **5.** $\frac{\pi}{3}$ **7.** $-\frac{\pi}{6}$

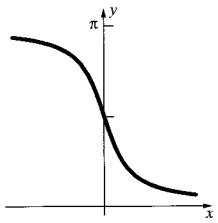
9. 0.4567 **11.** 0.1115 **13.** 0.9548

15. 2.038 **17.** 0.6259 **19.** $\theta = \sin^{-1}\frac{x}{8}$

21. $\theta = \sin^{-1}\frac{5}{x}$ **23.** $\theta = \tan^{-1}\frac{3}{x} - \tan^{-1}\frac{1}{x}$ **25.** $\frac{1}{9}$

27. $\frac{56}{65}$ **33. (a)** $\frac{\pi}{2}$; **(b)** $-\frac{\pi}{2}$

35.



37. $\sec x$ **39.** $\dfrac{4x}{\sqrt{1-4x^4}}$ **41.** $x^2\left[\dfrac{xe^x}{1+e^{2x}} + 3\tan^{-1}(e^x)\right]$

43. $\dfrac{3(\tan^{-1}x)^2}{1+x^2}$ **45.** $\dfrac{3}{|x|\sqrt{x^6-1}}$

47. $\dfrac{3(1+\sin^{-1}x)^2}{\sqrt{1-x^2}}$ **49.** $\frac{1}{4}\sin^2 2x + C$

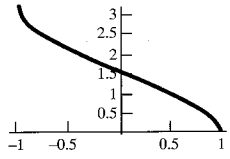
51. $\dfrac{\sin e^2 - \sin 1}{2}$ **53.** $\dfrac{\pi}{4}$

55. $\dfrac{\pi}{2}$ **57.** $\frac{1}{2}\arctan 2x + C$

59. $\theta = \tan^{-1}\dfrac{7.6}{b} - \tan^{-1}\dfrac{2.6}{b}$; If $b = 12.9, \theta \approx 0.3335$

63. $\pi b^2 - b^2\cos^{-1}\dfrac{b}{2a} - 2a^2\sin^{-1}\dfrac{b}{2a} + \dfrac{1}{2}b\sqrt{4a^2-b^2}$

65.



73. 4.9 ft **75.** $\frac{1}{13}$ rad/s **77.** 1 rev/min

79. 3.96×10^{-4} rad/s